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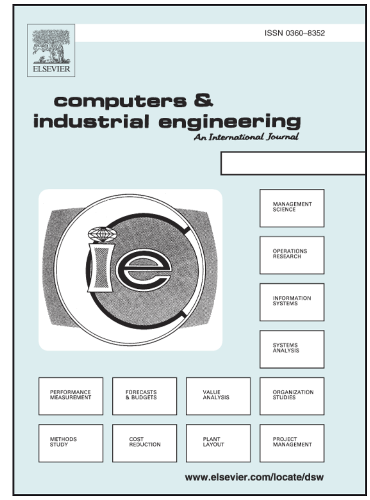

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Advance Selling with Double Marketing Efforts in a Newsvendor Framework

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Abstract: With the rapid development of e-commerce, advance selling has become a common practice in the retailing industry. In this paper, we look into a joint optimization problem of multiple and dynamic marketing decisions when advance selling is applied. We assume that a retailer sells a product with a short selling season. The product can be a perishable or deteriorating item. The retailer launches advance selling before the stock actually arrives at the market in order to extend the selling season and thereby increases the awareness of potential consumers to the new product. To further stimulate demand, the dual marketing efforts of advertising and price discounting are employed during the entire selling season. We first model and analyze the dynamic demand generating process based on an extension of Bass Diffusion Model (Bass, 1969). The paper also integrates two types of stochastic consumer valuation: the consumer value of the product and the loss of consumer value caused by advanced purchasing. Furthermore, we model the marketing decision as a deterministic Markov decision-making process and then develop the properties of the optimal solutions of the problem. In order to solve the model, a dynamic programming method is applied. At last, managerial insights are explored through a numerical study. Our study shows that prolonging the selling season with an advance selling season is an effective tool to improve sales performance, especially in combination with the mix of marketing efforts.

Keywords: Advance selling, Advertising, Price discount, Information diffusion, Dynamic programming, Newsvendor model

1. Introduction

Advance selling is a strategy that can enhance retailers' understanding of the market potential for a product and reduce demand uncertainty (Boyacı and Özer, 2010). It aims at attracting consumers to purchase earlier and increase spending. With the rapid development of e-commerce, advance selling has become prevalent in the online retailing industry. Almost all kinds of goods, including apparel, electronics, fresh food, and sports equipment, can be offered in advance selling. For example, the largest Chinese online retailing platform, Taobao.com, has launched 11th November (Single's Day) advance selling since 2009. Taobao.com usually starts advertising advance selling from 21st October. During the advance selling period, the retailers offer thrilling price discounts for advance orders and promise a delivery date that is normally after 11th November. The revenue of the Single's Day promotions has risen rapidly from no more than ¥0.1 billion in 2009 to ¥168.2 billion in 2017 in China (Xinhua Net, 2017).

Advance selling can be a natural extension of the normal selling season and helps the retailer reach out to a broader consumer base for goods with short selling seasons. However, Shugan and Xie (2000) show that the advantage of advance selling is not only to improve sales, but also to give the retailer better maneuvering control of operations. Furthermore, advance selling is an effective tool to help understand the market and facilitate pricing (Xie and Shugan, 2001), for forecasting (Moe and Fader, 2002), and capacity design (Boyacı and Özer, 2010).

During a product's selling season, the market and demand are usually developed and realized gradually over time. Advertising and price discounts are commonly used marketing tools to activate the potential market and induce consumer purchases. The degree of consumer awareness and the amount of final sales are often directly affected by the intensity of these marketing efforts.

In this paper, we extend the classic newsvendor model to consider an entire selling season that consists of an advance selling period in addition to the normal selling season, which is named the spot selling season in the rest of the paper. The retailer successively makes decisions about the dual marketing efforts, advertisements, and price discount. The advertisement expenditure and the price discount level are dynamically adjustable over the entire selling season. Moreover, the market of the product evolves dynamically over time, based on an information diffusion process and a consumer choice model. The information diffusion process is modeled by an adoption model for new products commonly used in marketing research: Bass Model(BM). In addition, we assume that consumers are heterogeneous with respect to the consumer value of the product and regarding the value loss of advanced purchasing. The demand is then modeled with the commonly used reservation price-based model.

We formulate the problem as a Markov decision process with continuous time and state transition. The objective is to maximize the total profit over the entire selling season by making the optimal dynamic advertisement, pricing, and order quantity decisions. The dual marketing efforts and the marketing dynamics cause the difficulties in solving the problem analytically and optimally. Therefore, we choose to discretize the decision-making horizon and then solve the model by a dynamic programming-based backward search algorithm. Eventually, several managerial insights are explored through numerical analyses.

Our study contributes to existing literature by: (1) providing a modeling framework and method to optimize with respect to the marketing dynamics; (2) exploring the value of integrating advance selling and order quantity decisions; (3) developing and applying a dynamic version of Bass' Model more suitable to fast changing online markets; (4) considering consumer heterogeneity and uncertainties in market demand; (5) integrating the effects of advertisement and word of mouth effects in the demand function; (6) solving the retailer's profit maximization problem by applying an effective dynamic programming algorithm.

The rest of the paper is organized as follows. In Section 2, we review the related research literature. In Section 3, the problem is described and the mathematical model is formulated. The structural properties of the model are addressed in Section 4. An algorithm to solve the model is provided in Section 5. In Section 6, a numerical study is presented. Finally, Section 7 contains conclusions and future research directions.

2. Literature review

As indicated above, the problem we are studying is rather complex. It is closely related to the separate research streams of advance selling, information diffusion, and joint optimization of advertising and pricing. For the sake of clarity, we review the related literature from the perspectives of these three different streams.

2.1 Advance selling

Advance selling has attracted a great deal of scholarly interest in the past decades. Numerous research results have been presented and published. The success of advance selling is rooted in service industries such as airlines and hotels where sellers deal with perishable products (Shugan and Xie, 2000). As advance selling has become more and more widespread in other industries, especially in online retailing, researchers have addressed the issue in more general settings. For example, Shugan and Xie (2005) explore the impact of competition on advance selling driven by consumer uncertainty about future consumption states. They show that advance selling can be a very effective marketing tool in a competitive setting. Boyaci and Ozer (2010) study information acquisition for capacity planning via pricing and advance selling. They focus on the mitigation effect of advance selling on demand uncertainty. Fay and Xie (2010) examine the general economics of purchase options and explore the differences in buyer uncertainty between advance selling and probabilistic selling. Zhao and Pang (2011) argue that demand uncertainty could favor a seller if the pricing mechanism is designed properly in advance selling. Advance selling helps to reduce demand uncertainty for retailers, but consumers may prefer not to make a purchase in the advance selling season unless an appropriate incentive is provided. Prasad et al. (2010) conclude that retailers should sell in advance if consumers' expected valuation exceeds their expected surplus.

However, all existing studies regarding advance selling seem to apply aggregate demand functions and ignore the marketing dynamics. In our study, we extend the existing literature by considering a dynamically evolving market affected by both advertising and price discounts. This emphasizes how advance selling expands the consumer base and stimulates demand.

2.2 Information diffusion

Information diffusion theory is often used to capture market dynamics. The literature on the information diffusion theory and its application in marketing dates back to Bass' model (Bass, 1969). Horsky and Simon (1983) modify the external effect parameter in Bass' model to be a function of advertising in order to examine the effects of advertising on the sales growth of a new product. Kalish (1985) characterizes the adoption of a new product by two steps: awareness and adoption, and analyzes optimal control of the diffusion process by pricing and advertising over time. Later, Bass et al. (1994) develop the General Bass Model (GBM) that incorporates dynamic marketing variables. Krishnan et al. (1999) and Krishnan and Jain (2006) use the GBM to derive an optimal pricing and advertising policy. Bass' model has also been modified and extended to deal with various other problems, such as in the Norton–Bass model by Norton and Bass (1987) and in the Piecewise-Diffusion Model by Niu (2006). For further details, see the review by Greenhalgh et al. (2004).

Furthermore, some studies focus on the information diffusion effects of advertising and pricing for new products. Bass' Model is thereby extended to a new-product adoption model. Sethi (1983) proposes a new-product adoption model with advertising, known as the Sethi model. Sethi et al. (2008) incorporate dynamic pricing and advertising effects into the Sethi model and use optimal control theory to determine the optimal dynamic marketing efforts. He et al. (2009) use the Sethi model to consider cooperative advertising and pricing in a dynamic stochastic supply chain. Helmes et al. (2013) generalize the model in Sethi et al. (2008) and derive optimal advertising and pricing policies. They take arbitrary adoption and saturation effects into account, and solve finite and infinite horizon-discounted variations of the associated control problems.

2.3 Joint optimization of advertising and pricing

It is common practice to apply multiple marketing efforts concurrently, for example, advertising and price discounts. Joint advertising and pricing problems have been studied extensively. Ray (2005) assumes that the consumer demand is sensitive towards both price and non-price factors (for example, advertisement), and determines the optimal pricing, stocking, and attribute level values in a newsvendor framework. Arcelus et al. (2006) evaluate the pricing and ordering policies of risk-neutral, risk-averse and risk-seeking newsvendor-type retailers and conclude that pricing, price discount and advertising are, in that order, the most profitable sales-promotion policies. Feng et al. (2014) propose a dynamic joint pricing and advertising optimization model to maximize total profit for perishable products. Chen (2015) evaluates the impact of price schemes and cooperative advertising mechanisms on dual-channel supply chain competition. In this literature, marketing efforts have a direct effect on the demand functions.

Raman and Chatterjee (1995) represent demand uncertainty by a Wiener process and examine the pricing policy for a monopolist in an uncertain demand environment. Kamrad et al. (2005) employ the same optimization framework to develop optimal pricing or advertising policies that maximize the value of innovation. While joint advertising and pricing problems have been studied extensively, the joint advertising, price promotion, and inventory optimization problem with advanced selling and market dynamics has not been studied previously. We aim at formulating and solving this problem in order to enrich the current literature and support industry practices.

Our study is most closely related to Horsky and Simon (1983), Kalish (1985), and Kamrad et al. (2005). We extend the Bass model addressed in Horsky and Simon (1983) to consider dynamic marketing effects on advertising and word of mouth. In accordance with Kalish (1985) and Kamrad et al. (2005), the information diffusion and the demand generation processes are characterized by two steps in the consumers' awareness and adoption behavior. Kamrad et al. (2005) assume that a marketing effort directly affects the informed consumers' purchase decisions. In our study, we extend the previous research by considering dual marketing efforts, advertising and price discounting, as well as the consumers' heterogeneity in valuation. Sequentially, the word-of-mouth and advertising effects first stimulate the consumer awareness level, and then the informed consumers make purchase decisions based on their individual valuations. Thus, the demand function is derived from both the information diffusion process and the consumer choice theory.

3. Problem description and formulation

We consider a retailer who introduces a new product with a short selling season to the market. In order to prolong the selling season and approach more consumers, the retailer may attempt advance selling before the spot selling season. With the rapid growth of online retail channels, dynamic advertisement and price discounting have become common marketing practices, and we therefore consider these dual marketing efforts by essentially extending the classic newsvendor model to include an advance selling season and a spot selling season.

During the advance selling season, the retailer needs to determine the intensity of advertisement and the advance purchase price discount over time. In the spot selling season, the retailer faces a newsvendor problem with a constant advertising intensity. We denote the time length of the advance selling season by T and the time length of the spot selling season by T_0.

We first present the main assumptions in order to formulate the problem as a tractable mathematical model.

Assumption 1: *The product's spot selling price is exogenous.*

This assumption is reasonable, because in the advance selling season the retailer needs to inform the consumers about the spot selling price of the product, so that they can make advance purchase decisions based on the offered price discount. The spot selling price is usually determined based on the current price of comparable products or historical sales data for similar products in the market. If the spot selling price is not pre-determined, consumers who purchase in the advance selling season are also likely to regard it as unfair if the spot selling price turns out to be lower than the price paid during the advance selling season.

Assumption 2*: The consumer has no awareness of the product at the beginning of the advance selling season.*

In case of a new product, consumers have no prior knowledge about the product before the start of the advance selling season. Advertising is used to inform the consumers about the product and related promotion activities. Thus, the sales realization process is characterized by two steps: awareness and adoption. "Awareness" refers to "activating a potential consumer via mass media or other adopters to be aware of the product"; and "adoption" refers to "making an advance order commitment", i.e., purchasing the product in the advance selling season (Kalish, 1985). Given the consecutive advertising and price discount offers, the sales realization process progresses dynamically.

In the following sub-sections, we analyze the problem based on the formulation of a mathematical model including a consumer awareness diffusion process, a demand (sales) realization process, and an integrated profit generation process.

For ease of exposition, the notations used in the paper are summarized in Table 1.

Table 1 Notation used in the paper

Parameters:	
b	The advertisement expense coefficient.
c	The procurement cost per unit of product for the retailer.
K	The consumer awareness effect of advertisement.
k	The consumer awareness sensitivity of advertisement.
p	The market price per unit of product for consumers in the spot selling season.
v	The random variable of consumer valuation, where $v \in [v_{min}, v_{max}]$, and v_{min} and v_{max} are the lower and upper limits.
β	The time-decay factor for the word-of-mouth effect.
Υ	The consumer awareness effect of word-of-mouth.
γ	The consumer awareness sensitivity of the word-of-mouth effect.
ω	The random variable of a consumer's value loss for purchases in the advance selling season, where $w \in [\omega_{min}, \omega_{max}]$, and ω_{min} and ω_{max} are the lower and upper limits.
State variables:	
i_t	The cumulative consumer awareness state, i.e., the proportion of informed consumers relative to market size at time t, $i_t \in [0,1)$.
s_t	The cumulative sales state, i.e., the proportion of cumulative sales realized over the potential market size at time t, $s_t \in [0,1)$.
ε	The random demand element in the spot selling season.
Decision variables:	
G_t	The intensity of advertisement at time t. $\frac{b}{2}G_t^2$ is the expenditure of advertisement.
Q	The quantity of products that the retailer orders in advance for the spot selling season.
z_t	The price discount at time t, $z_t \in [\underline{z}, \bar{z}]$, where $\underline{z}$ and $\bar{z}$ are the lower and upper limits, respectively, of the price discount.

3.1 The consumer awareness diffusion process

During the advance selling season, dynamic advertising and price discounting are applied. Advertising directly contributes to the consumer awareness about the product. Price discounting not only attracts more price sensitive consumers to make advanced purchases, but also stimulates consumers to convey the product information to others. This is usually named as the "word-of-mouth" effect.

The Bass Model assumes that potential adopters of a new product are influenced by both communication mass media (advertising) and word-of-mouth. The cumulative consumer awareness, i_t, is measured by the consumers who have become aware of the product in relation to the entire potential market. The growth curve of i_t is modeled as $\frac{di_t}{1-i_t} = [K + \Upsilon i_t]dt$, where di_t is the instant increase rate of consumer awareness at time t.

In the original Bass Model, the parameters, K and Υ, are constants, i.e., the consumer awareness effects of advertisement and word-of-mouth are static over time. However, this is no longer a realistic assumption given the realities of modern retailing practices when a retailer adjusts the marketing and pricing policies dynamically according to realized sales. Therefore, we modify the Bass Model to consider dynamic advertising and price discounting. Thus,

$$\frac{di_t}{1-i_t} = [K_t(G_t) + \Upsilon_t(z_t)i_t]dt. \tag{1}$$

where the advertisement effect at time t, $K_t(G_t)$, is a function of the advertisement intensity at time t, G_t. This captures the dynamically changing advertisement effect on the consumer awareness. The word-of-mouth effect at time t, $\Upsilon_t(z_t)$, is a function of the price discounts at time t, z_t. We consider an example of linear advertisement and word-of-mouth effects with the functional form $kG_t + \Upsilon_t z_t i_t$. In addition, we assume that the word-of-mouth effect decays over time with the decay factor, β. Thus, the cumulative consumer awareness can be modeled as

$$\frac{di_t}{dt} = \left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t),\ t \in [0, T]. \tag{2}$$

In the spot selling season, the price discount is no longer offered, but a constant advertisement intensity remains. Thus, the retailer faces a static advertising and order quantity decision problem. The cumulative consumer awareness in the spot selling season is modeled as

$$\frac{di_t}{dt} = kG_{T_0}(1 - i_t),\ t \in [T, T + T_0].$$

3.2 The demand realization process

After the consumers become aware of the product, some of them make advance purchase decisions to enjoy the discount, whereas others may wait until the product is actually available in the spot selling season. The consumers waiting until the spot selling season may try to purchase or just walk away. Thus, the demand realization process can be described by the decision tree in Figure 1.

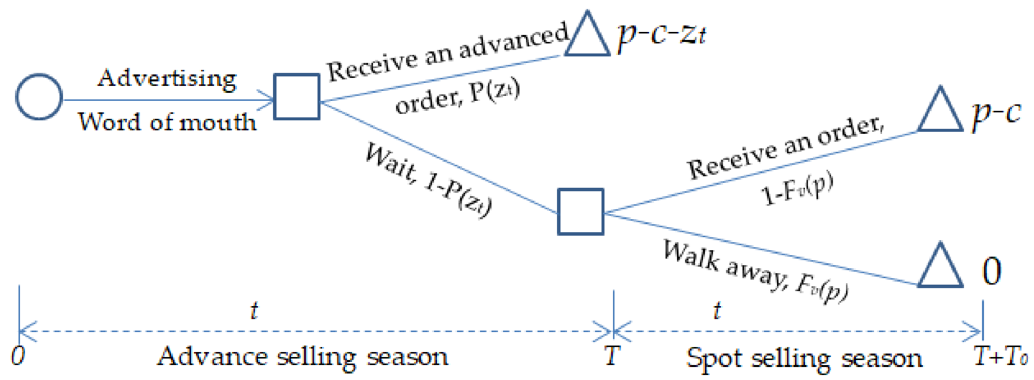

Figure 1 The demand realization process described as a decision tree

The consumer purchase decision is made based on the consumer value of the product and the value loss of an order. It is common that the consumers' value, v, is described by a uniformly distributed random variable with a p.d.f., f_v, and c.d.f., F_v, supported on the interval $[v_{min}, v_{max}]$. According to the classic consumer choice theory, consumers will buy at a price p only if their net utility, $v - p$, is non-negative (Prasad et al., 2010). Thus, in general, the expected purchase probability can be specified as $1 - F_v(p) = \int_p^{v_{max}} f_v dv$.

Because the potential market consists of all the informed consumers, it is therefore easy to estimate the expected aggregate demand function in the spot selling season as

$$E[D_{T_0}] = (i_{T+T_0} - s_T)(1 - F_v(p)).$$

where s_T is the cumulative sales state at the end of the advance selling season.

However, we may still have consumer behavior uncertainty to some extent, such as value changes, market fluctuations, etc. Thus, we consider a random demand element, ε, with $E(\varepsilon) = 0$ and $VAR(\varepsilon) = \sigma_\varepsilon{}^2$. The demand in the spot selling season, D_{T_0}, will be the consequence of $(i_{T+T_0} - s_T)$-fold Bernoulli trials, thus, $\sigma_\varepsilon{}^2 = (i_{T+T_0} - s_T)F_v(p)\big(1 - F_v(p)\big)$.

Hence, given the advance sales state at the end of the advance selling season and the consumer awareness state at the end of the spot season, we have the total demand in the spot selling season

$$D_{T_0}(i_T, s_T) = \big(i_{T+T_0} - s_T\big)(1 - F_v(p)) + \varepsilon. \tag{3}$$

In the advance selling season, the consumers' purchase decisions are affected not only by the original consumer valuation, v, but also by a value loss of advance ordering. The value loss of advance purchase is usually caused by risk perception regarding the quality of the product, advance commitment of payment, delayed delivery, etc. We consider a value loss, ω, supported by a uniform distribution on the interval $[\omega_{min}, \omega_{max}]$ with p.d.f, f_ω *and* c.d.f., F_ω.

Assumption 3: *The value loss of advance purchase, ω, is independent of the original consumer* *valuation,* v.

According to the consumer choice theory, when there are multiple purchase options, consumers will take the option with the highest positive net utility (Prasad et al., 2010). In addition, we make the following assumption.

Assumption 4: *The price discount is non-increasing during the advance selling season.*

In the spot selling season, consumers will purchase the product if $v - p \geq 0$. Analogously, in the advance selling season, the consumers who are aware of the product will buy it if $v - \omega - p + z_t \geq \max(v - p, 0)$. Consequently, the purchase probability at any time t during the advance selling season is

$$\mathrm{P}(p - z_t) = \mathrm{Prob}(v - \omega - p + z_t > \max(v - p, 0)) \tag{4}$$

Furthermore, when $v - p < 0$, a consumer who is aware of the product during the advance selling season will buy it only if $v - \omega - p + z_t > 0$; on the other hand, when $v - p \geq 0$, the consumer will buy the product during either the advance selling season or the spot selling season. If the value loss is less than the price discount, i.e., if $\omega \leq z_t$ the consumer will make the purchase during the advance selling season. Therefore, Eq. (4) can be rewritten as

$$\mathrm{P}(p - z_t) = \mathrm{Prob}\{(\omega \leq z_t |\, v - p \geq 0) \text{ or } (v - \omega - p + z_t > 0 \,|v - p < 0)\}$$

According to Assumption 3, the consumer valuation, v, is independent of the value loss of an advance purchase, ω. Hence, we have

$$\mathrm{P}(p - z_t) = F_\omega(z_t)\big(1 - F_v(\,p)\big) + \mathrm{Prob}(v - \omega - p + z_t > 0 \,|v - p < 0), \tag{5}$$

where the first term is a linear function of z_t,

$$F_\omega(z_t)\big(1 - F_v(\,p)\big) = \frac{z_t - \omega_{min}}{\omega_{max} - \omega_{min}}\big(1 - F_v(\,p)\big).$$

and the second term is

$$Prob\ (v - \omega - p + z_t > 0 \,|v - p < 0) = \int_{v_{min}}^{p} \int_{\omega_{min}}^{v-p+z_t} f_{v,\omega}(x, y)\, dxdy.$$

According to the convolution of two independent uniform random variables, we have,

$$f_{v,\omega}(x, y) = \begin{cases} \dfrac{1}{(v_{max} - v_{min})(\omega_{max} - \omega_{min})}, & v_{min} \leq x \leq v_{max}, \omega_{min} \leq y \leq \omega_{max} \\ 0 & else \end{cases}$$

Under different conditions of parameters $v, \omega, and\ p$, the second term can be further written as:

Case 1: When $p - \omega_{max} \leq v_{min} - \omega_{min}$,

$Prob\ (v - \omega - p + z_t > 0 \,|v - p < 0) =$

$$f_{v,\omega}(x,y)\begin{cases}\frac{(z_t-\omega_{min})^2}{2}, & \omega_{min}\le z_t\le p-v_{min}+\omega_{min}\\ (p-v_{min})z_t-\frac{(p-v_{min})^2}{2}, & p-v_{min}+\omega_{min}<z_t<\omega_{max}\\ \frac{(p-v_{min})}{v_{max}-v_{min}}-\frac{(p-z_t+w_{max}-v_{min})^2}{2}, & \omega_{max}\le z_t\le p-v_{min}+\omega_{max}\end{cases}\tag{6a}$$

Case 2: When $p-\omega_{max}>v_{min}-\omega_{min}$,

$Prob\,(v-\omega-p+z_t>0\,|v-p<0)=$

$$f_{v,\omega}(x,y)\begin{cases}\frac{(z_t-\omega_{min})^2}{2}, & \omega_{min}\le z_t\le\omega_{max}\\ \frac{(\omega_{max}-\omega_{min})^2}{2}+(z_t-\omega_{max})(\omega_{max}-\omega_{min}), & \omega_{max}<z_t<p-v_{min}+\omega_{min}\\ (p-v_{min})(\omega_{max}-\omega_{min})-\frac{(p-v_{min}+\omega_{max}-z_t)^2}{2}, & p-v_{min}+\omega_{min}\le z_t\le p-v_{min}+\omega_{max}\end{cases}\tag{6b}$$

It is clear that $\frac{dP\,(v-\omega-p+z_t>0\,|v-p<0)}{dz_t}\ge 0$ for all segments in both cases, i.e.,

$$\frac{d\mathrm{P}(p-z_t)}{dz_t}=\frac{1-F_v(\,p)}{\omega_{max}-\omega_{min}}+\frac{dP\,(v-\omega-p+z_t>0\,|v-p<0)}{dz_t}\ge 0.$$

In both Eqs. (6a) and (6b), for the first segment,

$$\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2}=f_{v,\omega}(x,y)\ge 0;$$

for the second segment,

$$\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2}=0;$$

and for the third segment,

$$\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2}=-f_{v,\omega}(x,y)\le 0.$$

In addition, at time $t\le T$, the cumulative advance sales state, s_t, is realized, and the consumer awareness state, i_t, can also be estimated. Therefore,

$$ds_t=\mathrm{P}(p-z_t)di_t,\;\; t\le T\tag{7}$$

3.3 Profit generation process

The objective of the retailer is to maximize the expected profit over the entire selling season. Let $\boldsymbol{G}=\{G_t,t\in[0,T+T_0]\}$ and $\boldsymbol{z}=\{z_t,t\in[0,T+T_0]\}$. The expected profit can be formulated as

$$max\,\Pi(\boldsymbol{G},\boldsymbol{z},Q)=\pi_T+\pi_{T_0}(i_T,s_T),\tag{8}$$

where π_T is the expected profit generated from the advance selling season, i.e.,

$$\pi_T = max \int_0^T \left(\left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t)\mathrm{P}(p - z_t)(p - c - z_t) - \frac{b}{2} {G_t}^2 \right) dt \, . \tag{9}$$

The advertising expenditures at time t are specified as $\frac{b}{2}{G_t}^2$. A quadratic cost function is common in the literature (Jørgensen, 2000 and He et al., 2009).

$\pi_{T_0}(i_T, s_T)$ is the expected profit generated in the spot selling season, i.e.,

$$\pi_{T_0}(i_T, s_T) = \max_{G_t, Q} E[pD_{T_0} - cQ - max\,(0, D_{T_0} - Q)\,p - \int_T^{T+T_0} \frac{b}{2} {G_t}^2 dt], \tag{10}$$

where

$$D_{T_0}(i_T, s_T) = \left(i_{T+T_0} - s_T\right)\left(1 - F_v(p)\right) + \varepsilon$$

$$= \left(\int_0^T \left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t)\left(1 - \mathrm{P}(p - z_t)\right)dt + \int_T^{T+T_0} kG_t(1 - i_t)dt\right)\left(1 - F_v(p)\right) + \varepsilon.$$

During the advance selling season, orders are continuously collected and sold at the unit price $p - z_t$. After the advance selling season, when the number of units sold, s_T, is known, the retailer orders $s_T + Q$ units from the supplier. Before the spot selling season begins, the complete order arrives at the retailer's warehouse and all units sold as advance orders are delivered first. The order size Q will then cover the uncertain demand D_{T_0} during the spot selling season when the units are sold at the fixed price p for immediate delivery, as long as stock is available. We assume that the lead time is negligible compared to the two selling seasons or that there is a frozen period for the lead time between the two selling seasons.

4. Structural results

In this section, we analyze the property of the revenue function structure and the conditions of the optimal solution.

4.1 The property of the revenue function

A consumer who becomes aware of the product during the advance selling season generates an expected marginal net revenue

$$R(z_t) = (p - c)(1 - F_v(p)) + \mathrm{P}(p - z_t)((p - c)F_v(p) - z_t),$$

where $(p - c)(1 - F_v(p))$ is the spot selling net revenue, and $\mathrm{P}(p - z_t)((p - c)F_v(p) - z_t)$ is the advance selling revenue, and $(p - c)F_v(p)$ is the loss of net revenue when an informed consumer walks away. Thus, a price discount would be profitable only if $z_t \leq (p - c)F_v(p)$. Furthermore, a useful property of the revenue function, $R(z_t)$ is explored.

Property of $\boldsymbol{R(z_t)}$: When $p \leq \frac{(2v_{max} + 2\omega_{min} + cF_v(p))}{(F_v(p) + 2)}$, the revenue function, $R(z_t)$, is concave. When $p > \frac{(2v_{max} + 2\omega_{min} + cF_v(p))}{(F_v(p) + 2)}$, the revenue function, $R(z_t)$, is quasi-concave.

Proof: First, since $\mathrm{P}(p - z_t)$ is continuous, the revenue function, $R(z_t)$, is also continuous.

Moreover,

$$\frac{dR(z_t)}{dz_t} = \frac{d\mathrm{P}(p-z_t)}{dz_t}\big((p-c)F_v(p) - z_t\big) - \mathrm{P}(p-z_t)\,,$$

and

$$\frac{d^2R(z_t)}{dz_t^2} = \frac{d^2\mathrm{P}(p-z_t)}{dz_t^2}\big((p-c)F_v(p) - z_t\big) - 2\frac{d\mathrm{P}(p-z_t)}{dz_t}.$$

Concavity of the revenue function can be analyzed based on the three segments of the purchase probability function, $P(p-z_t)$. We present the proof using the case in Eq. (6a).

(1) For the first segment of Eq. (6a), we have $\frac{d^2P(p-z_t)}{dz_t^2} = f_{v,\omega}(x,y)$ and $\frac{dP(p-z_t)}{dz_t} = [(v_{max}-p)+(z_t-\omega_{min})]f_{v,\omega}(x,y)$. Thus,

$$\frac{d^2R(z_t)}{dz_t^2} = f_{v,\omega}(x,y)[(p-c)F_v(p) - 3z_t + 2\omega_{min} - 2(v_{max}-p)]$$

If $z_t \geq \frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3}$ and $w_{min} \leq z_t < p - v_{min} + w_{min}$, then, $\frac{d^2R(z_t)}{dz_t^2} \leq 0$. In more details, we have,

a. If $w_{min} \leq \frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3} \leq p - v_{min} + w_{min}$, then if $\frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3} \leq z_t < p - v_{min} + w_{min}$, we obtain $\frac{d^2R(z_t)}{dz_t^2} \leq 0$;

b. If $w_{min} \geq \frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3}$, then if $w_{min} \leq z_t < p - v_{min} + w_{min}$, we obtain $\frac{d^2R(z_t)}{dz_t^2} \leq 0$;

c. If $w_{min} \leq z_t < \frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3} < p - v_{min} + w_{min}$, then $\frac{d^2R(z_t)}{dz_t^2} > 0$. However, this condition is rarely satisfied, unless the retail price, p, is rather high.

Since if $\frac{(p-c)F_v(p)+2\omega_{min}-2(v_{max}-p)}{3} > 0$, it requires that $p > (2v_{max} + 2\omega_{min} + cF_v(p))/(F_v(p)+2)$. This means that $R(z_t)$ is concave in the third segment of Eq. (6a) when $p \leq (2v_{max} + 2\omega_{min} + cF_v(p))/(F_v(p)+2)$.

In addition, since $\frac{dR(z_t=0)}{dz_t} = \frac{dP(p-z_t)}{dz_t}(p-c)F_v(p) > 0$ and $\frac{dR(z_t)}{dz_t} > \frac{dR(z_t=0)}{dz_t} > 0$, it shows that $R(z_t)$ is increasing in z_t in the third segment of Eq. (6a).

(2) For the second segment in Eq. (6a), we have $\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2} = 0$ and $\frac{d\mathrm{P}(p-z_t)}{dz_t} \geq 0$, then $\frac{d^2R(z_t)}{dz_t^2} \leq 0$. Therefore, the revenue function $R(z_t)$ is always concave when $p - v_{min} + w_{min} \leq z_t < w_{max}$. Moreover, the continuity and increasing property of $R(z_t)$ proved above in the first segment shows that $R(z_t)$ is at least quasi-concave, when $p > (2v_{max} + 2\omega_{min} + cF_v(p))/(F_v(p)+2)$.

(3) For the third segment of Eq. (6a), we have $\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2} = -f_{v,\omega}(x,y) \leq 0$ and $\frac{d\mathrm{P}(p-z_t)}{dz_t} =$

$\frac{(1-F_v(p))}{w_{max}-w_{min}} + (p - v_{min} + \omega_{max} - z_t) f_{v,\omega}(x, y) \geq 0.$

Thus, if $z_t \leq (p-c)F_v(p)$ in addition to the segment range of z_t, $\omega_{max} \leq z_t < p - v_{min} + \omega_{max}$, then $\frac{d^2R(z_t)}{dz_t^2} \leq 0$. In more details,

a. If $\omega_{max} \leq (p-c)F_v(p) \leq p - v_{min} + \omega_{max}$, then $\frac{d^2R(z_t)}{dz_t^2} \leq 0$, where $\omega_{max} \leq z_t \leq (p-c)F_v(p)$;

b. If $(p-c)F_v(p) \geq p - v_{min} + \omega_{max}$, then $\frac{d^2R(z_t)}{dz_t^2} \leq 0$, where $\omega_{max} \leq z_t < p - v_{min} + \omega_{max}$.

If $(p-c)F_v(p) \leq z_t < p - v_{min} + \omega_{max}$, when $\frac{d^2\mathrm{P}(p-z_t)}{dz_t^2}\big((p-c)F_v(p) - z_t\big) \leq 2\frac{d\mathrm{P}(p-z_t)}{dz_t}$, we may still have $\frac{d^2R(z_t)}{dz_t^2} \leq 0$. By rearranging the inequality, we obtain the condition,

$z_t \geq -\frac{1}{2}\left[p - v_{min} + \omega_{max} + (p-c)F_v(p) + \frac{(1-F_v(p))}{(w_{max}-w_{min})f_{v,\omega}(x,y)}\right]$. Since $p - v_{min} + \omega_{max} + (p-c)F_v(p) + \frac{(1-F_v(p))}{(w_{max}-w_{min})f_{v,\omega}(x,y)} > 0$ and $z_t \geq w_{min}$, the condition $z_t \geq -\frac{1}{2}\left[p - v_{min} + \omega_{max} + (p-c)F_v(p) + \frac{(1-F_v(p))}{(w_{max}-w_{min})f_{v,\omega}(x,y)}\right]$ always holds. Therefore, the revenue function $R(z_t)$ is always concave when $\omega_{max} \leq z_t < p - v_{min} + \omega_{max}$.

In addition, $\frac{dR(z_t)}{dz_t} = \frac{d\mathrm{P}(p-z_t)}{dz_t}\big((p-c)F_v(p) - z_t\big) - \mathrm{P}(p-z_t) < 0$, i.e., $R(z_t)$ is decreasing in z_t.

Therefore, the conclusion holds. Analogously, the concavity of the revenue function under the case of Eq. (6b) can also be proved. ■

With the given model parameters, the concavity property of the revenue function, $R(z_t)$, can easily be determined. The numerical study also shows that the revenue function $R(z_t)$ is actually overall concave when $p \leq \frac{(2v_{max}+2\omega_{min}+cF_v(p))}{(F_v(p)+2)}$. We use Figure 2 to illustrate the result in Property of $R(z_t)$. Figure 2(a) is set with normal price, $p = 150$ for numerical instances (1-15) in Table 2 and 2(b) is set with extreme price, $p = 190$ for instances (16-19).

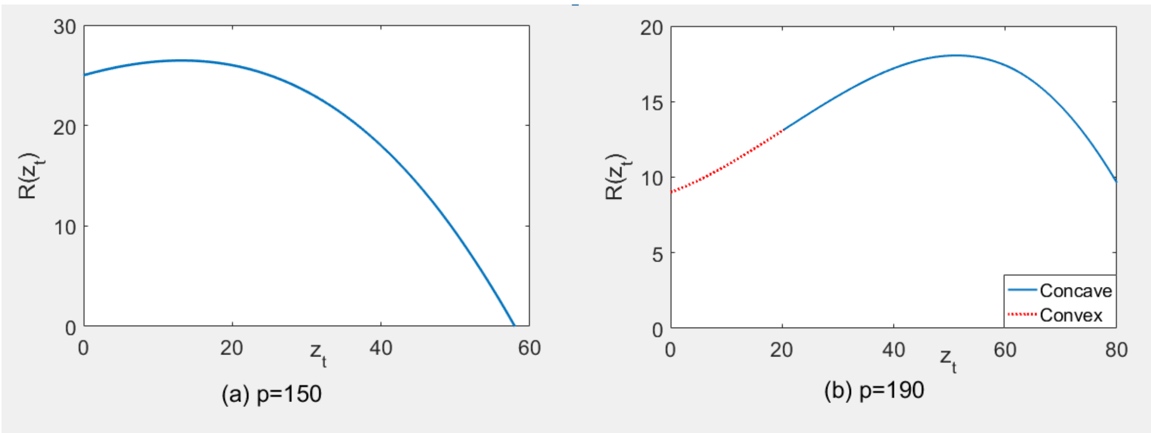

Figure 2. Illustration of the concavity of $R(z_t)$

Fig 2(a) shows that the revenue function is overall concave for $p = 150 \le (2v_{max} + 2\omega_{min} + cF_v(p))/(F_v(p) + 2) = 180$. Fig (2b) shows that the revenue function is quasi-concave when $p = 190 > (2v_{max} + 2\omega_{min} + cF_v(p))/(F_v(p) + 2) = 169$.

When $R(z_t)$ is concave (or quasi-concave) based on the first-order condition, $dR(z_t)/dz_t = 0$, we can find the stationary point, z_t^0, so that the marginal net revenue is maximal at time t. Moreover, if the price discount z_t is bounded by the interval $[\underline{z}, \bar{z}]$, the optimal instant solution of the price discount, z_t^*, will be found in one of the following three cases.

1) If $z_t^0 \le \underline{z}$, then $z_t^* = \underline{z}$. This results from a very low spot price and a great spot purchase probability, $1 - F_v(p)$. The retailer consequently has an incentive to advertise the product extensively.

2) If $z_t^0 \ge \bar{z}$, then $z_t^* = \bar{z}$. This results from a very high spot price and a small spot purchase probability, $1 - F_v(p)$. The price discount in the advance selling season plays an important role in maximization of total profit.

3) If $z_t^0 \in (\underline{z}, \bar{z})$, then $z_t^* \in (\underline{z}, \bar{z})$. In the following, our analysis will focus on this case.

Based on the analysis of the consumer purchase behavior above, we now reformulate the profit function by integrating profits realized both in the spot selling season and in the advance selling season. Eqs. (8)-(10) are rewritten as

$$max\ \Pi(G, z, Q) = \pi_T' + \pi_{T_0}'(i_T), \tag{8'}$$

where the profit in the advance selling season is

$$\pi_T' = \max_{G_t, z_t} E[\int_0^T \left((kG_t + \gamma e^{-\beta t} z_t i_t)(1 - i_t)R(z_t) - \frac{b}{2} {G_t}^2 \right) dt], \tag{9'}$$

and the profit in the spot selling season is

$$\pi_{T_0}'(i_T) = \max_{G_t, Q} E[pD_{T_0}' - cQ' - \max(0, D_{T_0}' - Q')p - \int_T^{T+T_0} \frac{b}{2} {G_t}^2 dt], \tag{10'}$$

where $D_{T_0}' = (i_{T+T_0} - i_T)(1 - F_v(p)) + \varepsilon$ and $Q' = Q - (i_T - s_T)(1 - F_v(p))$.

The demand function, D'_{T_0}, is no longer the demand in the spot selling season, D_{T_0}. The demands from consumers informed during the advance selling season but who purchased during the spot selling season are now accounted for as demands of the advance selling season. Correspondingly, Q' is to cover D'_{T_0}, the demand awared and realized in the spot selling season. Thus, the demand realized but informed in advanced selling season is,

$$\Delta D = D_{T_0} - D'_{T_0} = Q - Q' = (i_T - s_T)\big(1 - F_v(p)\big)$$

$$= \big(1 - F_v(p)\big)\int_0^T \big(kG_t + \gamma e^{-\beta t} z_t i_t\big)(1 - i_t)\big(1 - \mathrm{P}(p - z_t)\big)dt.$$

The objective function in Eq. (8′), the total profit of the retailer, is non-linear and continuous, which causes the difficulty in solving the problem to optimality in general. Considering the two separate selling seasons, we first analyze the problem in the spot selling season and then consider the problem in the advance selling season.

4.2 Optimal solutions in the spot selling season

In the spot selling season, the company faces a newsvendor problem with an added decision of advertisement intensity. Given a consumer awareness state at the end of the advance selling season, i_T, the advertisement intensity and order quantity in the spot selling season can be determined explicitly. As described in Subsection 3.1, we simplify the problem by introducing the following assumption.

Assumption 5: *During the spot selling season, the retailer uses a static advertisement policy denoted by* G_{T_0}.

The static advertising policy is reasonable because the spot selling season is usually relatively short, and the price discount is no longer offered during this season. Dynamic advertising may not be possible or necessary.

From Eq. (1), we now have

$$\frac{di_t}{(1-i_t)} = kG_{T_0}dt.$$

Taking the definite integral of both sides, we obtain

$$\ln\big(1 - i_{T+T_0}\big) - \ln(1 - i_T) = kT_0 G_{T_0}.$$

Therefore, given the cumulative consumer awareness state, i_T at the end of time T, the expected level of consumer awareness at the end of the entire selling season will be

$$i_{T+T_0} = 1 - (1 - i_T)e^{-kT_0 G_{T_0}}.$$

Then the demand will be

$$D'_{T_0} = (1 - (1 - i_T)e^{-kT_0 G_{T_0}} - i_T)(1 - F_v(p)) + \varepsilon.$$

Furthermore, the expected profit function in the spot selling season is

$$\pi'_{T_0}(i_T) = \max_{G_{T_0},Q} E[(1-(1-i_T)e^{-kT_0G_{T_0}} - i_T)(1-F_v(p))p - cQ' - max(0, D'_{T_0} - Q')p - \frac{b}{2}T_0{G_{T_0}}^2]$$

Solving the newsvendor model, the optimal order quantity is

$$Q'^* = (1-(1-i_T)e^{-kT_0G_{T_0}} - i_T)(1-F_v(p)) + \Phi^{-1}(\tfrac{p-c}{p})\sigma_\varepsilon.$$

Moreover, the expected profit function is

$$\pi'^*_{T_0}(i_T) = (p-c)(1-(1-i_T)e^{-kT_0G_{T_0}} - i_T)(1-F_v(p)) - p\phi(\Phi^{-1}(\tfrac{p-c}{p}))\sigma_\varepsilon - \tfrac{bT_0}{2}G^2_{T_0} \quad (11)$$

Furthermore, since $\frac{d^2\pi'_{T_0}(Q'^*)}{dG^2_{T_0}} = -(1-i_T)e^{-kT_0G^*_{T_0}}(kT_0)^2(p-c)(1-F_v(p)) - bT_0 \leq 0$, given the optimal order quantity Q'^*, the optimal solution of advertising intensity, $G^*_{T_0}$ is unique and satisfies the first-order-condition

$$e^{-kT_0G^*_{T_0}} = \frac{bG^*_{T_0}}{k(1-i_T)(p-c)(1-F_v(p))} \quad (12)$$

Eq. (12) can be solved by simple numerical methods.

4.3 Optimal conditions in the advance selling season

In the advance selling season, the retailer can dynamically change the advertising and price discount decisions over time t so that the total expected profit is maximized. At any time t, given a consumer awareness state, i_t, the expected value function, the profit-to-go, is

$$V_t(i_t) = \max_{\boldsymbol{G},\boldsymbol{z}}\{\int_t^T \left(R(z_\tau)(kG_\tau + \gamma e^{-\beta t}z_\tau i_\tau)(1-i_\tau) - \tfrac{b}{2}{G_\tau}^2\right)d\tau + \pi'^*_{T_0}(i_T, T)\} \quad (13)$$

Obviously, the standard deviation of the random factor, σ_ε, plays little influence on advance selling decision. Hence, we define simplified value function in the spot selling season instead of $\pi'^*_{T_0}(i_T, T)$ in Eq.(11),

$$V_{T_0}(i_T, T) = (p-c)(1-(1-i_T)e^{-kT_0G_{T_0}} - i_T)(1-\mathrm{F}_v(p)) - \tfrac{bT_0}{2}G^2_{T_0},$$

which removes term, $-p\phi(\Phi^{-1}(\frac{p-c}{p}))\sigma_\varepsilon$ from $\pi'^*_{T_0}(i_T, T)$, and does not change the property and characteristic of our models. Thus, Eq. (13) can be written as

$$V_t(i_t) = \max_{\boldsymbol{G},\boldsymbol{z}}\{\int_t^T \left[R(z_\tau)(kG_\tau + \gamma e^{-\beta t}z_\tau i_\tau)(1-i_\tau) - \tfrac{b}{2}{G_\tau}^2\right]d\tau + V_{T_0}(i_T, T)\} \quad (13')$$

Given the optimal solutions in terms of order quantity, Q'^*, and advertising intensity, $G^*_{T_0}$, in the spot selling season, the HJB equation is

$$HV_t = \max_{G_t,z_t} \begin{matrix}\{R(z_t)(kG_t + \gamma e^{-\beta t}z_t i_t)(1-i_t) - \tfrac{b}{2}{G_t}^2 \\ +V_i(kG_t + \gamma e^{-\beta t}z_t i_t)(1-i_t)\}\end{matrix} \quad (14)$$

where V_i is the gradient of the value function in the state variable, i.e., $V_i = \frac{\partial V^*_t(i_t)}{\partial i_t}$.

The optimal solutions of our joint advertisement and price discount model are the optimal solutions to maximize the HJB equation (Dockner, 2000). Therefore, we focus on analyzing the HJB equation in Eq. (14).

From the first-order conditions, we obtain

$$\frac{\partial HV_t}{\partial G_t} = k(R(z_t) + V_i)(1 - i_t) - bG_t = 0 \tag{15}$$

$$\frac{\partial \mathrm{HV}_t}{\partial z_t} = \frac{dR(z_t)}{dz_t}\left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t) + (R(z_t) + V_i)\gamma e^{-\beta t} i_t(1 - i_t) = 0 \tag{16}$$

Substituting Eq. (15) into Eq. (16), V_i is eliminated, and we have

$$\frac{dR(z_t)}{dz_t}\left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t) + \frac{bG_t\gamma e^{-\beta t} i_t}{k} = 0 \tag{17}$$

Rearranging Eq. (17), we obtain

$$G_t^*(z_t^*) = -\frac{\frac{dR(z_t^*)}{dz_t^*}\gamma e^{-\beta t} z_t^* i_t(1-i_t)}{\frac{dR(z_t^*)}{dz_t^*}k(1-i_t) + \frac{b\gamma e^{-\beta t} i_t}{k}},\ t \in (0, T] \tag{18}$$

Theoretically, we can take the feedback solutions into the state transition function in Eq. (2) and obtain the optimal trajectory of i_t. However, in general, it is difficult to solve Eq. (2) exactly. Therefore, we analyze the optimal solution structure and then resort to dynamic search algorithms.

Proposition 1: When the revenue function $R(z_t)$ is concave in z_t, the optimal solutions of our original problem satisfy the condition in Eq. (18) or it is obtained at points z_t^0 or $\bar{z}$, where z_t^0 is the stationary point specified by $dR(z_t^0)/dz_t^0 = 0$, so that the marginal revenue is maximal at time t.

Proof: Solving the equation system (15)-(16), we obtain the stationary points of the HJB equation, denoted as $(z_t^*, G_t^*(z_t^*))$, $t \in (0, T]$. Substituting Eq. (18) into Eq. (16), we find that the stationary points, z_t^*, that satisfy the condition in Eq. (17).

According to the second-order conditions, we have

$$A = \frac{\partial^2 \mathrm{HV_t}}{\partial {G_t}^2} = -b < 0$$

$$B = \frac{\partial^2 \mathrm{HV_t}}{\partial G_t\, \partial z_t} = \frac{dR(z_t)}{dz_t} k(1 - i_t)$$

$$C = \frac{\partial^2 \mathrm{HV_t}}{\partial {z_t}^2} = \frac{d^2R(z_t)}{d{z_t}^2}\left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t) + 2\frac{dR(z_t)}{dz_t}\gamma e^{-\beta t} i_t(1 - i_t)$$

Hence,

$$AC - B^2 = -b\frac{d^2R(z_t)}{d{z_t}^2}\left(kG_t + \gamma e^{-\beta t} z_t i_t\right)(1 - i_t) - \frac{dR(z_t)}{dz_t}(1 - i_t)[\frac{dR(z_t)}{dz_t}k^2(1 - i_t) + 2b\gamma e^{-\beta t} i_t]$$

The optimal solutions should be in one of the following two cases:

(1) If $z_t^* \in (z_t^0, \bar{z}]$, we have $\frac{dR(z_t^*)}{dz_t^*} \leq 0$, and thus, the term $\frac{dR({z_t}^*)}{d{z_t}^*}\gamma e^{-\beta t} {z_t}^* i_t(1 - i_t) \leq 0$ in

Eq. (17). This requires that $\frac{dR(z_t^*)}{dz_t^*}kG_t(1-i_t)+\frac{bG_t\gamma e^{-\beta t}i_t}{k}\geq 0$, so that the solution of Eq. (17) exists. Thus, $\frac{dR(z_t^*)}{dz_t^*}k^2(1-i_t)+2b\gamma e^{-\beta t}i_t>0$ is established.

According to Property of $R(z_t)$, $\frac{d^2R(z_t)}{d{z_t}^2}\leq 0$, and therefore, $AC-B^2>0$ at z_t^*. According to the extreme value theorem, the stationary points $(z_t^*, G_t^*(z_t^*))$ are maximum value points. Therefore, if $z_t^0<z_t^*<\bar{z}$, the optimal solution should be z_t^*; on the other hand, if $z_t^*>\bar{z}$, the optimal solution should be $\bar{z}$.

(2) If $z_t\in[\underline{z}, z_t^0]$, we have $\frac{dR(z_t)}{dz_t}\geq 0$, and from Eqs. (15) and (16), $\frac{\partial \mathrm{HV}_t}{\partial z_t}=\frac{dR(z_t)}{dz_t}\big(kG_t+\gamma e^{-\beta t}z_ti_t\big)(1-i_t)+\frac{bG_t\gamma e^{-\beta t}i_t}{k}>0$. Therefore, V_t is an increasing function in z_t when $z_t<z_t^0$. Hence, when the stationary points are in the range $[\underline{z}, z_t^0]$, the optimal solution should always be z_t^0. ■

We have established the structural properties to characterize the optimal solutions. However, we still have no explicit method for obtaining the optimal solutions of the original problem in continuous decision time. Therefore, based on Proposition 1, we discretize the continuous control problem into a multi-period problem and propose a dynamic programming algorithm to optimize the advertisement and the price discount over the entire selling season.

5. Dynamic programming algorithm

We discretize the problem into a multi-period problem. The continuous decision planning horizon is split into finite periods, $t=\{1,2,\cdots,T,T+1\}$. The length of each period is denoted by L_t. We rewrite the continuous decision model as a backward dynamic programming model.

According to Eq. (7) in sub-section 3.2, the sales state variable s_t depends on the consumer awareness state variable i_t. According to Proposition 1, the optimal advertisement intensity strategy variable depends on the price discount strategy. Therefore, we describe the dynamic programming process only based on the consumer awareness state variable and the price discount strategy variable.

The state space is $I=[0,1]$, and the consumer awareness state variable at the beginning of the period t is $i_t\in I$, $0\leq i_{t-1}\leq i_t\leq 1$, $t=1,\cdots,T$. The initial consumer awareness state is i_1. Thus, i_t can also be understood as the consumer awareness state at the end of period $t-1$.

The strategy space is $\Lambda=[\underline{z},\bar{z}]$, and the price discount strategy is $z_t\in\Lambda$ and $z_t\leq p$.

Furthermore, the state transition function is

$$i_{t+1}=i_t+L_t\big(kG_t(z_t)+\gamma e^{-\beta t}z_ti_t\big)(1-i_t),\ t=1,2\cdots,T+1 \tag{19}$$

where $i_{T+2} = i_{T+T_0}$.

The retailer's value function at the end of period t is

$$V_t(i_t) = \max_{z_t} E[(i_{t+1} - i_t)R(z_t) - \frac{b}{2} L_t {G_t}^2(z_t) + V_{t+1}(i_{t+1})], \; t = 1,2 \cdots, T \tag{20}$$

where the value function for the spot selling period is

$$V_{T+1}(i_{T+1}) = V_{T_0}(i_{T+1}) \tag{21}$$

At the end of the entire selling season, the value of the product becomes zero. Thus, we have the boundary condition $V_{T+2}(i_{T+2}) = 0$.

The dynamic programming model described above is equivalent to a deterministic Markov process. The existence of an optimal solution is shown in Lemma 1.

Lemma 1 (Puterman, M. L., 1994): Given the finite state space, I, and the finite strategy space, Λ , the original problem is a deterministic Markov decision-making process and an optimal Markovian policy exists.

Proof: According to the problem analysis in Section 3, we know that the state transition process is deterministic, so the problem is a deterministic Markovian process. The consumer awareness is contained in [0,1]. Hence, the state space $I = \{i, 0 \le i \le 1,\}$ is finite. The joint marketing effect of advertisement and price discount is $kG_t + \gamma e^{-\beta t} z_t i_t \le 1$, and therefore, $kG_t \le 1$, so $0 \le G_t < \frac{1}{k}$. In addition, the decision space z_t is subject to $0 \le \mathrm{P}(p - z_t) \le 1$, and $R(z_t) > 0$, so the decision space is compact. Therefore, an optimal Markovian policy exists. ■

Therefore, the problem can be solved by a backward dynamic programming algorithm as specified below.

The Algorithm

Step 0. Let $t = T + 1$ *[Representing the spot selling season]*

Step 0.1 Initialize i_t and increment step size Δi, and let $k = 1$;

Step 0.2 Determine the optimal solutions of $G_t^{k^*}(i_t^k)$ according to Eq. (12); record and calculate the profit function $V_t^{k^*}(i_t^k)$=$\pi'_{T'}(i_t^k)$;

Step 0.3 Let $k = k + 1$ and ${i_{t-1}}_t^k = i_{t-1} + \Delta i$. If $i_t < 1$, go to step 0.2; otherwise go to step 1;

Step 1 Let $t = T$ *[Representing the end of the advance selling season]*

Step 1.1 Initialize the consumer awareness state i_t and the step size Δi, and let $k = 1$ and $i_t^k = i_t$;

Step 1.2 According to Proposition 1, solve the value function (20) and find the optimal solution $(G_t^{k^*}(z_t^{k^*}), z_{t+1}^{k^*}(i_t^k))$; then calculate and record the maximum profit function, $V_t^{k^*}(i_t^k, z_t^k)$, and the consumer awareness state transit to $i_{t+1}^k(z_t^*, G_t^*)$;

Step 1.3 Let $i_t^k = i_t^k + \Delta i$, and $k = k + 1$. If $i_t^k < 1$, go to Step 1.2; otherwise, go to Step 2.

Step 2 Let $t = t - 1$. If $t > 1$, go to step 1.1; otherwise, go to Step 3

Step 3 For $t = 1$. Initiate the total profit of the firm $\Pi = 0$; [Initiating t*he forward process to retrieve the dynamic solutions.]*

Step 3.1 Based on the known initial awareness state i_1, find and record the corresponding price discount and optimal advertisement, $z_1^*(i_1)$ and $G_1^*(z_1^*)$, respectively. Compute and record the state at the beginning of period 2, $i_2(z_1^*)$, and the total profit of the firm at the end of period 1 as $\pi_1(z_1^0)$. Thus, $\Pi = \Pi + \pi_1(z_1^*)$; proceed to Step 3.2;

Step 3.2 Let $t = t + 1$. If $t \leq T + 1$, go to Step 3.3; otherwise, stop.

Step 3.3 Determine the optimal consumer awareness state $i_t^* = argmaxV_t(i_t)$; find and record the corresponding price discount and optimal advertisement, $z_t^*(i_t^*)$ and $G_t^*(z_t^*)$, respectively. Let $\Pi = \Pi + V_t^*(i_t^*)$; go to Step 3.2.

When the decision period lengths during the advance selling season approach zero, the optimal solution from the dynamic programming algorithm approaches the optimal solution of our original continuous time model. In addition, the dynamic programming algorithm also provides a state-dependent solution for any period to maximize the profit-to-go. Thus, it can be modified to accommodate the demand uncertainties by a re-optimization method.

6. Numerical study

In this section, we apply numerical instances to test the algorithm and verify the value of the advance selling and double marketing efforts. We employ two special cases, a "Non-discount" case that involves only advertisement efforts and a "Non-advance selling"case that is the classic newsvendor model with advertising, as benchmarks.

A set of hypothetical instances are constructed by varying the parameter values. The purchase cost and retail price are $c = 100$, $p = 150$. We assume $b = 2$ and $\beta = 1$. The consumer value is uniformly distributed as $v \sim U[100, 200]$. The consumer value loss for advanced purchase is uniformly distributed as $\omega \sim U[0,60]$. A new product is launched, and the initial consumer awareness state is $i_1 = 0.1$. The spot selling season of the product is two weeks. The firm now also implements a four-week advance selling period.

We consider the advertising sensitivity, *k*, at three levels, $\{10\%, 20\%, 30\%\}$. The word-of-mouth sensitivity, γ, can be at five different levels, $\{1\%, 2\%, 3\%, 4\%, 5\%\}$. The word-of-mouth has highly varying influence in various scenarios and hence we choose to consider more levels of the word-of-mouth sensitivity.

Hence, 15 instances are generated based on different k and γ values. We set the parameters in a limited range that favors the integrated advanced selling and spot selling format. Under smaller k or γ than those considered, the advance selling case completely dominates the non-advance selling case. Under greater k or γ than those considered, the spot selling season dominates the sales, and advance selling becomes less effective.

We assume that the advance selling season can be split into four periods. There are five periods in total with a spot selling period. We solve the model by the dynamic programming algorithm in Section 5. The optimal solutions are presented in Table 2.

The columns (3)-(7) show the optimal solutions on the advertising, price discount, and profit, respectively. Column (8) is the maximal profit when no discount is offered, and the last column (9) shows the maximal profit when the advance selling is not implemented. All solutions of advertising G_t and profits are calculated based on the number of potential consumers for the whole market base while the price discount is the discount from the normal price.

First, the numerical results show a significant improvement in the profit of the retailer when advance selling is implemented, compared to the normal newsvendor problem with advertising. In addition, price discount is effective in advance selling season to improve profit. In fact, the advance selling policy improves the sales performance in two ways: a direct profit improvement from price discounts and an indirect profit enhancement by diffusing information to a larger consumer base.

Table 2 Optimal solutions of numerical instances

	Parameters		Advertise	Discount				Profit	
	k	γ	$(G_1, G_2, G_3, G_4, G_5)$	(z_1, z_2, z_3, z_4)	π_T	$\pi_{T'}$	Π	NO DISCOUNT	NO ADVANCE
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)
1	10%	1%	(0.9,0.8,0.8, 0.8,0.7)	(17.5,20.3,17.7,15.6)	-0.54	9.07	8.53	7.45	4.59
2		2%	(0.9,0.8,0.7, 0.7,0.6)	(21.3,25.0,22.5,18.7)	0.20	9.12	9.32		
3		3%	(1.0,0.7,0.6, 0.7,0.6)	(23.9,28.6,26.7,21.5)	0.75	9.56	10.31		
4		4%	(1.,0.7,0.6, 0.6,0.5)	(26.0,30.3, 28.9,25.1)	1.32	10.15	11.47		
5		5%	(1.0, 0.6, 0.5,0.5, 0.5)	(27.6,31.7,31.0,28.1)	2.03	10.71	12.74		
6	20%	1%	(1.0,0.9,0.9, 0.9 0.7)	(15.0,16.9,15.8,14.7)	-0.21	14.60	14.39	13.10	8.25
7		2%	(1.1,0.9,0.9, 0.9, 0.6)	(16.5,20.1,18.4,16.4)	0.07	14.83	14.90		
8		3%	(1.2, 0.9, 0.8, 0.8, 0.6)	(18.0,22.8,21.6,18.6)	0.60	14.90	15.50		
9		4%	(1.2, 0.9, 0.7, 0.7, 0.5)	(19.4,24.7,24.5,21.1)	1.25	14.95	16.20		
10		5%	(1.3, 0.8, 0.6, 0.6, 0.5)	(20.4,26.6,27.0,23.8)	1.77	15.20	16.97		
11	30%	1%	(1.0,0.9,0.9, 0.9, 0.6)	(14.1,15.6,15.1,14.4)	0.39	17.61	18.00	16.59	11.37
12		2%	(1.1,0.9,0.8, 0.8, 0.6)	(15.0,18.2,17.6,15.9)	0.86	17.49	18.35		
13		3%	(1.2,0.9,0.8, 0.7, 0.5)	(15.7,20.4,19.9,18.0)	1.15	17.63	18.78		
14		4%	(1.2,0.9,0.7, 0.6, 0.5)	(16.6,22.2,22.5, 20.3)	1.71	17.54	19.25		
15		5%	(1.3, 0.8, 0.6, 0.5, 0.4)	(17.3,24.3,25.1,23.0)	2.15	17.64	19.79		
				Special cases ($p = 190$)					
16	10%	1%	(0.8,0.6,0.6, 0.6,0.3)	(55.4,58.2,56.1,54.0)	3.19	2.51	5.70	1.71	1.20
17		5%	(0.6,0.4,0.2, 0.2,0.1)	(59.7,61.1,62.8,63.1)	9.43	4.48	13.91		
18	30%	1%	(1.0,0.9,0.7, 0.7,0.3)	(52.3,53.9,53.9,52.9)	6.55	4.93	11.48	4.37	2.71
19		5%	(0.8,0.6,0.2, 0.1,0.1)	(54.4,58.1,61.7,62.8)	10.43	5.54	15.97		

In addition, we considered four additional instances (instances 16-19) with a high price near v_{max} to show the impact of the non-concave part of the revenue function, $R(z_t)$ (see Figure 2). All parameters remain the same as in the instances 1-15 apart from the retail price, which is $p = 190$. For these instances, we search the entire decision space for the optimal solutions. The results show that the profits are much lower than those of the corresponding instances with p=150, since the market potential of the product has been dramatically reduced. These instances show that an extremely high price is not competitive in the market.

The computational process is rather complex. To help illustrate the algorithm, we explain the detailed calculation steps in the Appendix for Instance 1.

Moreover, our model is mainly based on a dynamic information diffusion process. Information diffusion results are determined by the word-of-mouth effect and the advertising effect. The coefficients of the word-of-mouth effect and the advertising effect are normally obtained through marketing research in each individual industry or company. However, the marketing research part is beyond the scope of our study. Therefore, we have conducted a sensitivity analysis to explore the sensitivity of the optimal solutions regarding the word-of-mouth effect and the advertising effect.

We look into the sensitivity of total sales and total profit to the coefficients γ and k. With respect to the numerical instances in Table 2, we consider variation ranges of $\pm$50% with a 10% increment. The sales and profit curves regarding the change of word-of-mouth effects are presented in Fig 3. As expected, the results show that the profit and sales are increasing in the word-of-mouth effect. However, with higher coefficients of the advertising effect, both profit and sales become less sensitive than in the cases with lower coefficients of advertising effect. Furthermore, when we observe the variation rates of profit and sales in Fig 4, the range of change is relatively small, about $\pm$5%.

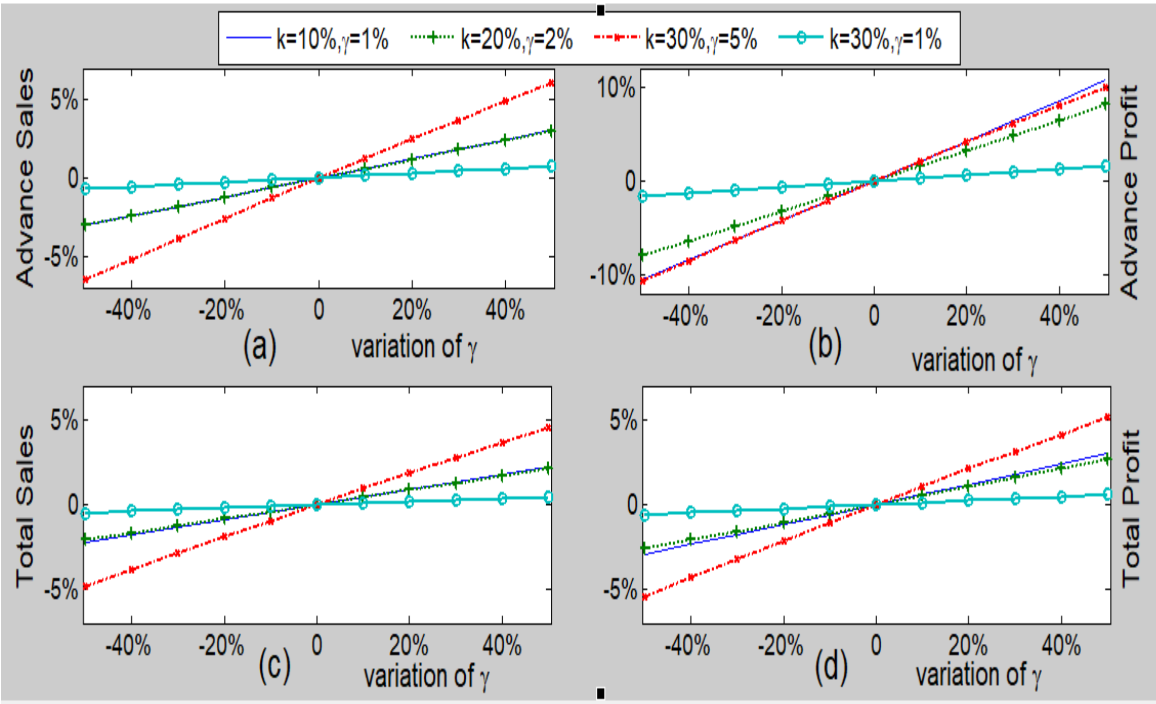

Fig 3. (a) sensitivity of advance sales to word-of-mouth effect; (b) sensitivity of advance selling profit to word-of-mouth effect; (c) sensitivity of total sales to word-of-mouth effect; (d) sensitivity of total profit to word-of-mouth effect.

The sales and profit curves regarding the change of advertising effects are shown in Fig 4. The results show, as expected, that the profit and sales are also increasing in the advertising effect. Observing the variation rates of profit and sales in Fig 4, the range of change is relatively large, about $\pm 20\%$. The reason is, however, that the advertising effect coefficient is larger than the word-of-mouth effect coefficient. In addition, according to Table 2, when increasing k by 50%, the optimal advertisement solutions do not change dramatically (column (3)). This means that the advertisement expenditure is not very sensitive to k either.

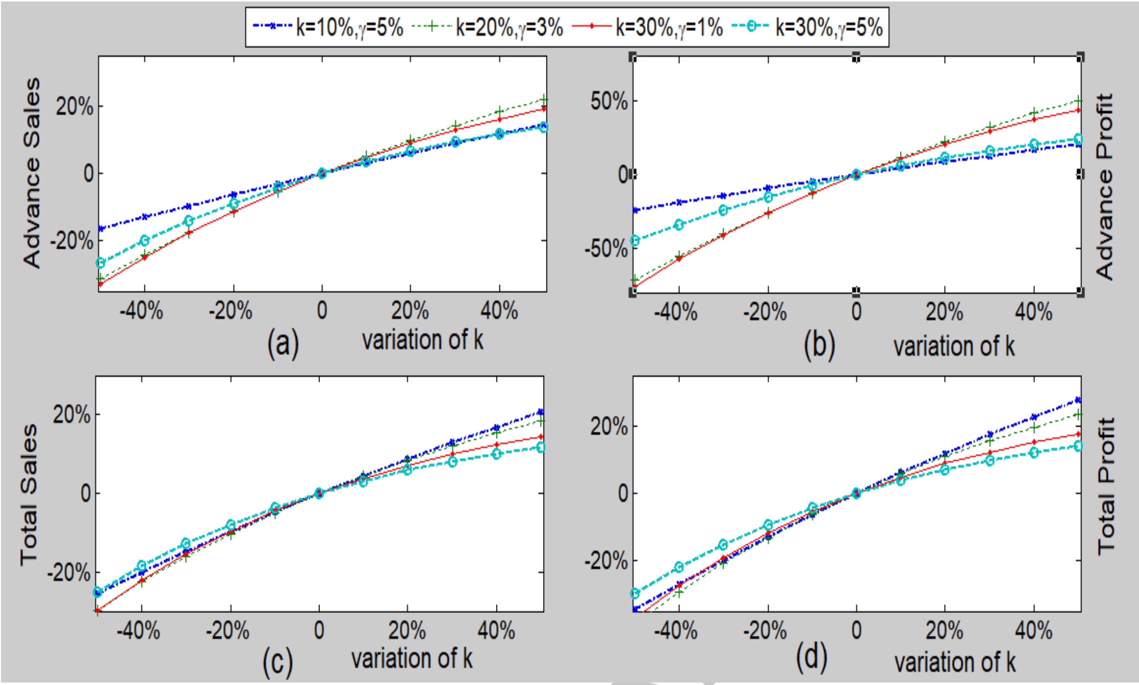

Fig 4. (a) sensitivity of advance sales to advertising effect; (b) sensitivity of advance selling profit to advertising effect; (c) sensitivity of total sales to advertising effect; (d) sensitivity of total profit to advertising effect.

7. Conclusion and future research

In this paper, we have investigated a dynamic optimization problem of advertising and pricing. We extend the classic Newsvendor model with advertisement and include advance selling. The demand generation process of a new product is related to dynamic information diffusion and consumer choice behavior. Consumer heterogeneities are considered by the stochastic consumer value and value loss of advance purchase. The retailer makes double marketing efforts by including advertising and price discounting. We have shown that advance selling can create much more profit for the retailer than in the cases with no advance selling and no discounting. We have shown that advance selling not only brings profit in the advance selling season, but also creates profit in the spot selling season, because advance selling expands the consumer awareness level, i.e., the market size. Thus, it is valuable to implement advance selling in order to improve the information diffusion effects and stimulate consumer incentives to commit to advance orders. Given the complex demand generation process and the stochastic factors, the model becomes difficult to solve. Therefore, we first strive to explore the structural problem properties. Based on these properties, we propose an effective algorithm to solve the model.

In the current research, we have assumed that the advertisement and word-of-mouth effects are pre-known and constant. It would be interesting to look for empirical evidence on how to estimate these effects for different types of products. It would also be interesting to investigate and test our model in real-world business cases. In addition, we have assumed that the consumer awareness accumulation process is deterministic, but in reality, it would quite likely be stochastic.

Appendix: Computational procedure of an instance

With regard to Instance 1 in Table 2, $k = 10\%$, and $\gamma = 1\%$. Given the consumer value distribution, $v \sim U[100, 200]$, we get $1 - F_v(p) = 0.5$.

Furthermore, according to Eq. (6a) and $\omega \sim U[0,60]$, we obtain,

$$\mathrm{P}(p - z_t) = \begin{cases} \frac{z_t^2}{12000} + \frac{z_t}{120} & 0 \le z_t < 50 \\ \frac{z_t}{60} - \frac{5}{24} & 50 \le z_t < 60 \\ \min\left(1, -\frac{(110-z_t)^2}{12000} + \frac{z_t}{120} + \frac{1}{2}\right) & 60 \le z_t \le 110 \end{cases}$$

Then, according to Eq. (11), the marginal revenue function is obtained,

$$R(z_t) = 25 + \mathrm{P}(p - z_t)(25 - z_t)$$

When $0 \le z_t < 50$, $\frac{dR(z_t)}{dz_t} = -\frac{z_t^2}{4000} - \frac{z_t}{80} + \frac{5}{24}$.

Therefore, according to Eq. (16), we have $e^{-0.2G_{T_0}^*} = \frac{G_{T_0}^*}{1.25(1-i_T)}$. We can get $G_{T_0}^*$ and $V_{T_0}(i_T,)$ based on any given i_T.

Moreover, Eq. (18) becomes

$$G_t^*(z_t^*) = -\frac{dR(z_t^*)/dz_t^*\, z_t^* i_t(1-i_t)}{10R(z_t^*)/dz_t^*(1-i_t)e^t + 20i_t} \tag{A1}$$

Substituting the condition in Eq. (A1) into the value function (20), we have

$$V_t^*(i_t, z_t) = \max_{z_t} E[(kG_t(z_t) + \gamma z_t i_t)(1 - i_t)(25 + \mathrm{P}(p - z_t)(25 - z_t)) - e^{T-t}L_t b {G_t}^2(z_t)/2 + V_{t+1}(i_{t+1})],$$
$$t = 1, \cdots, T + 1 \tag{A2}$$

where the $V_{T+1}(i_{T+1})$ is the value function in the spot selling period. We can see that the function is only related to the consumer awareness states at the beginning of each period and the price discount decision variable at each period. Therefore, we can solve the model by the standard backward dynamic programming method.

We start from the spot selling season. Given any possible consumer awareness state, i_{T+1}, discretized by percentages of $\{1\%, 2\%, \cdots, 99\%, 100\%\}$, we calculate the optimal spot advertisement intensity according to Eq. (12) to obtain $G_{T+1}^{k*}(i_{T+1}^k)$, and then record the corresponding expected spot profits $E[V_{T+1}^{k*}] = 25\left(1 - \left(1 - i_{T+1}^k\right)e^{-0.1G_{T+1}^{k*}} - i_{T+1}^k\right) - 2{G_{T+1}^{k*}}^2$.

Next, at each advance selling period t, $t = T, \cdots, 1$, given any consumer awareness state i_t, we search for the optimal solution of price discount, z_t^{k*}. The detailed calculation data from the dynamic programming procedure is presented in Tables A.1-A.5. Finally, we retrieve the optimal state trajectory is $\{i_1 = 10\%,\ i_2 = 20\%,\ i_3 = 27\%,\ i_4 = 34\%,\ i_5 = 39\%,$

$i_6 = 47\%\}$ and the optimal solutions showed in Table 2. Meanwhile, with the realized advance sales, $s_4 = 0.07$, we estimate $\sigma_\varepsilon = 0.31$. Then the optimal order quantity is $Q^* + s_4 = 0.137$.

Table A.1 Dynamic programming computation table when $t = 5$

State i_5	G_5	V_5
20%	0.8446	1.6818
21%	0.8355	1.6431
22%	0.8265	1.6048
23%	0.8173	1.5669
24%	0.8082	1.5294
25%	0.799	1.4923
26%	0.7898	1.4556
27%	0.7806	1.4192
28%	0.7713	1.3833
29%	0.762	1.3478
30%	0.7527	1.3126
31%	0.7433	1.2779
…	…	…
70%	0.3497	0.262

Table A.2 Dynamic programming computation table when $t = 4$

i_4	z_4	G_4	i_5	Profit in t_4	Profit to go	V_4
20%	14.44	0.6106	25%	0.9499	1.4923	2.4422
	14.45	0.7356	26%	1.0461	1.4556	2.5017
	14.45	0.8606	27%	1.1111	1.4192	2.5303
	14.45	0.9856	28%	1.1449	1.3833	2.5282
	14.45	1.1106	29%	1.1474	1.3478	2.4952
21%	14.52	0.6177	26%	0.941	1.4556	2.3966
	14.52	0.7443	27%	1.0331	1.4192	2.4523
	14.53	0.8709	28%	1.0932	1.3833	2.4765
	14.53	0.9975	29%	1.1212	1.3478	2.469
	14.53	1.124	30%	1.1172	1.3126	2.4298
22%	14.6	0.625	27%	0.9318	1.4192	2.351
	14.6	0.7532	28%	1.0196	1.3833	2.4029
	14.61	0.8814	29%	1.0746	1.3478	2.4224
	14.61	1.0096	30%	1.0966	1.3126	2.4092
	14.61	1.1378	31%	1.0858	1.2779	2.3637
…	…	…	…	…	…	…
50%	17.09	0.1575	51%	0.2384	0.671	0.9094
	17.58	0.3562	52%	0.3987	0.6453	1.044
	17.75	0.5558	53%	0.479	0.62	1.099
	17.83	0.7556	54%	0.4793	0.5951	1.0744

17.88	0.9555	55%	0.3996	0.5707	0.9703

Table A.3 Dynamic programming computation table when $t = 3$

i_3	z_3	G_3	i_4	Profit in t_3	Profit to go	V_3
15%	15.49	0.5568	20%	1.0109	2.5303	3.5412
	15.51	0.6744	21%	1.1302	2.4765	3.6067
	15.52	0.792	22%	1.2218	2.4224	3.6442
	15.53	0.9097	23%	1.2858	2.3678	3.6536
	15.54	1.0273	24%	1.3221	2.3129	3.635
16%	15.68	0.6903	22%	1.1217	2.4224	3.5441
	15.7	0.7993	23%	1.2096	2.3678	3.5774
	15.71	0.9184	24%	1.2692	2.3129	3.5821
	15.72	1.0374	25%	1.3004	2.2576	3.558
	15.73	1.1564	26%	1.3033	2.2035	3.5068
17%	15.83	0.566	22%	0.9997	2.4224	3.4221
	15.85	0.6864	23%	1.1128	2.3678	3.4806
	15.87	0.8069	24%	1.1969	2.3129	3.5098
	15.89	0.9273	25%	1.252	2.2576	3.5096
	15.9	1.0478	26%	1.278	2.2035	3.4815
…	…	…	…	…	…	…
45%	18.32	0.0702	46%	0.256	1.2188	1.4748
	21.4	0.2333	47%	0.4491	1.181	1.6301
	22.61	0.4078	48%	0.5635	1.1431	1.7066
	23.26	0.5856	49%	0.5978	1.105	1.7028
	23.67	0.7649	50%	0.5515	1.0665	1.618

Table A.4 Dynamic programming computation table when $t = 2$

i_2	z_2	G_2	i_3	Profit in t_2	Profit to go	V_2
9%	16.54	0.6046	15%	1.216	3.6536	4.8696
	16.58	0.7143	16%	1.3347	3.5821	4.9168
	16.61	0.8241	17%	1.4292	3.5098	4.939
	16.63	0.934	18%	1.4995	3.4413	4.9408
	16.65	1.0438	19%	1.5457	3.3736	4.9193
10%	16.82	0.4937	15%	1.0733	3.6536	4.7269
	16.89	0.6045	16%	1.2146	3.5821	4.7967
	16.94	0.7155	17%	1.3313	3.5098	4.8411
	16.98	0.8264	18%	1.4232	3.4413	4.8645
	17.01	0.9374	19%	1.4905	3.3736	4.8641
	…	…	…	…	…	…
…	22.83	0.1647	41%	0.7286	1.9057	2.6343
38%	25.48	0.289	42%	0.8822	1.8563	2.7385
	27.34	0.4243	43%	0.9731	1.806	2.7791
	28.7	0.5665	44%	0.9984	1.7549	2.7533

	29.74	0.7133	45%	0.9565	1.7066	2.6631
	17.62	0.601	15%	1.0889	3.4264	4.5153

Table A.5 Dynamic programming computation table when $t = 1$

i_1	z_1	G_1	i_2	Profit in t_1	Profit to go	V_1
	16.04	0.341	9%	2.2018	4.9168	7.1186
	16.51	0.444	10%	2.3423	4.8411	7.1834
5%	16.89	0.547	11%	2.4502	4.7907	7.2409
	17.21	0.651	12%	2.5253	4.7198	7.2451
	17.48	0.755	13%	2.5676	4.6479	7.2155
	16.71	0.432	11%	2.6202	4.7907	7.4109
	17.16	0.535	12%	2.7301	4.7198	7.4499
6%	17.55	0.639	13%	2.8066	4.6479	7.4545
	17.88	0.744	14%	2.8498	4.5749	7.4247
	18.17	0.848	15%	2.8595	4.5006	7.3601
	16.84	0.42	12%	2.8977	4.7198	7.6175
	17.37	0.524	13%	3.0096	4.6479	7.6575
7%	17.81	0.628	14%	3.0876	4.5749	7.6625
	18.2	0.733	15%	3.1317	4.5006	7.6323
	18.54	0.838	16%	3.1418	4.4242	7.566
…	…	…	…	…	…	…
	16.56	0.277	27%	7.0023	3.3896	10.3919
	17.62	0.382	28%	7.1397	3.3306	10.4703
26%	18.59	0.489	29%	7.2356	3.2756	10.5112
	19.5	0.597	30%	7.2893	3.2195	10.5088
	20.35	0.706	31%	7.3003	3.164	10.4643

In the computation tables, we only present part of the computational data. In our numerical study, the solutions are obtained by using a search step size of 1%. Even if the problem includes only four advanced periods, the computation amount is quite large. For the sake of brevity, we only present part of the computational tables here. During the optimal solution search process, we found that the value function is in fact uni-modal. Therefore, we only present the state space and solutions close to the optimal solutions. However, we actually search the entire state and strategy spaces.

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Highlights:

1) Advance selling is an effective tool to prolong selling season and improve the selling performance.
2) A dynamic version of Bass Model is developed and applied to advance selling.
3) Effective dynamic programming method is applied to solve the model to optimality.